Carpenter, T.P., Fennema, E., Franke, M.L., Levi, L., & Empson, S.B. (1999). Addition and subtraction: Children's solution strategies. In *Children's Mathematics: Cognitively Guided Instruction* (15-31). Portsmouth, NH: Heinemann.

3 ADDITION AND SUBTRACTION
Children's Solution Strategies

Research has identified a reasonably coherent picture of the strategies that children invent to solve addition and subtraction problems and how they evolve over time. The distinctions among problem types are reflected in children's solution processes. For the most basic strategies, children use physical objects (counters) or fingers to directly model the action or relationships described in each problem. Over time, children's strategies become more abstract and efficient. Direct Modeling strategies are replaced by more abstract Counting strategies, which in turn are replaced with number facts.

DIRECT MODELING STRATEGIES

Children invent Direct Modeling strategies to solve many of the problem types discussed earlier. To solve Join (Result Unknown) or Part-Part-Whole (Whole Unknown) problems, they use objects or fingers to represent each of the addends, and then they count the union of the two sets. We illustrate this strategy, called *Joining All*, in the following example:

> *Robin had 4 toy cars. Her friends gave her 7 more toy cars for her birthday. How many toy cars did she have then?*

> Karla makes a set of 4 cubes and a set of 7 cubes. She pushes them together and then counts them, "1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11," pointing to a cube with each count. Karla then responds, "She had 11 cars."

A similar strategy is used to solve Join (Change Unknown) problems. The primary difference is that the goal is to find the number of objects added to the initial set rather than the total. The child makes a set equivalent to the initial quantity and adds objects to it until the new collection is equal to the total given in the problem. The number of objects added is the answer. This strategy, called *Joining To*, is illustrated in the following example and in Figure 3.1:

> *Robin has 4 toy cars. How many more toy cars does she need to get for her birthday to have 11 toy cars all together?*

> Karla makes a set of 4 cubes. She adds additional cubes, counting, "5, 6, 7, 8, 9, 10, 11," until there is a total of 11 cubes. She keeps the cubes that she adds separate from the initial set of 4 cubes so that she can count them separately. She then counts the 7 cubes. Karla responds, "She needs 7 more."

One important difference between this strategy and the Joining All strategy is that children must somehow be able to distinguish the counters that they join to the initial set from the counters in the initial set so that they can count them separately. They may do this by keeping the counters physically separate or by using differently colored counters. This requires some advanced planning that the Joining All strategy does not.

The strategy that best models the Separate (Result Unknown) problem involves a subtracting or separating action. In this case, the larger quantity in the problem is initially represented, and the smaller quantity is subsequently removed from it. We give an example of this strategy, called *Separating From*, below:

> *Colleen had 12 guppies. She gave 5 guppies to Roger. How many guppies does Colleen have left?*

FIGURE 3.1
Using a Joining To Strategy to Solve a Join (Change Unknown) Problem

Karla makes a set of 12 cubes and removes 5 of them. She counts the remaining cubes. Karla then responds, "She has 7 left."

The Separate (Change Unknown) problem also involves a separating action. The strategy generally used to solve this problem is similar to the Separating From strategy except that objects are removed from the larger set until the number of objects remaining is equal to the smaller number given in the problem. The following example illustrates this strategy, called *Separating To*:

Roger had 13 stickers. He gave some to Colleen. He has 4 stickers left. How many stickers did he give to Colleen?

Karla makes a set of 13 cubes. She slowly removes cubes one by one, looking at the cubes remaining in the initial set. When she has removed 6 cubes, she counts the cubes in the remaining set. Finding that she has 7 cubes left, she removes 3 more cubes and again counts the cubes in the remaining set. Finding that there are now 4 cubes left, she stops removing cubes and counts the 9 cubes that were removed. Karla then responds, "He gave her 9."

Separating To involves a certain amount of trial and error because the child can't simply count objects as they are physically removed but must check the initial set to determine whether the appropriate number of objects remains. Children can most easily apply this strategy with small numbers so that the child can directly perceive whether there are two or three objects left.

Compare (Difference Unknown) problems describe a matching process. The strategy used to solve these problems involves the construction of a one-to-one correspondence between two sets until one set is exhausted. Counting the unmatched elements gives the answer. We illustrate the *Matching* strategy in the following example and in Figure 3.2 on page 18:

Mark has 6 mice. Joy has 11 mice. Joy has how many more mice than Mark?

Carl counts out a set of 6 cubes and another set of 11 cubes. He puts the set of 6 cubes in a row. He then makes a row of the 11 cubes next to the row of 6 cubes so that 6 of the cubes are aligned with the 6 cubes in the initial row. He then counts the 5 cubes that are not matched with a cube in the initial row. Carl responds, "She has 5 more."

It is difficult to model the Start Unknown problems because the initial quantity is unknown and therefore cannot be represented. A few children attempt to solve these problems using *Trial and Error*.

FIGURE 3.2
Using a Matching Strategy to Solve a Compare (Difference Unknown) Problem

The following example shows one such attempt:

Robin had some toy cars. Her friends gave her 5 more toy cars for her birthday. Then she had 11 toy cars. How many toy cars did Robin have before her birthday?

Karla counts out 3 cubes. She then adds 5 cubes to the original set and counts the total. Finding that the total is 8 rather than 11, she puts the cubes back with the unused cubes and starts over. Next she makes a set of 5 cubes and adds 5 more to it. Again she counts and realizes her original estimate is too low. This time she appears to recognize that she is only off by 1, so she adds 1 to her original set of 5 and then joins the other set of 5 to it. Counting the total, she finds that it is now 11. She recounts the first set of 6 cubes. She responds, "She had 6 before her birthday."

This example of Trial and Error illustrates a reasonably systematic attempt to solve the problem. When the first two estimates were too low, Karla increased them. Some children who attempt Trial and Error are less systematic.

Figure 3.3 summarizes the six Direct Modeling strategies described above.

COUNTING STRATEGIES

Counting strategies are more efficient and abstract than modeling with physical objects. In applying these strategies, a child recognizes that it is not

FIGURE 3.3
Direct Modeling Strategies

Problem	Strategy Description
Join (Result Unknown) Ellen had 3 tomatoes. She picked 5 more tomatoes. How many tomatoes does Ellen have now?	*Joining All* A set of 3 objects and a set of 5 objects are constructed. The sets are joined and the union of the two sets is counted.
Join (Change Unknown) Chuck had 3 peanuts. Clara gave him some more peanuts. Now Chuck has 8 peanuts. How many peanuts did Clara give him?	*Joining To* A set of 3 objects is constructed. Objects are added to this set until there is a total of 8 objects. The answer is found by counting the number of objects added.
Separate (Result Unknown) There were 8 seals playing. 3 seals swam away. How many seals were still playing?	*Separating From* A set of 8 objects is constructed. 3 objects are removed. The answer is the number of remaining objects.
Separate (Change Unknown) There were 8 people on the bus. Some people got off. Now there are 3 people on the bus. How many people got off the bus?	*Separating To* A set of 8 objects is counted out. Objects are removed from it until the number of objects remaining is equal to 3. The answer is the number of objects removed.
Compare (Difference Unknown) Megan has 3 stickers. Randy has 8 stickers. How many more stickers does Randy have than Megan?	*Matching* A set of 3 objects and a set of 8 objects are matched 1-to-1 until one set is used up. The answer is the number of unmatched objects remaining in the larger set.
Join (Start Unknown) Deborah had some books. She went to the library and got 3 more books. Now she has 8 books altogether. How many books did she have to start with?	*Trial and Error* A set of objects is constructed. A set of 3 objects is added to the set, and the resulting set is counted. If the final count is 8, then the number of objects in the initial set is the answer. If it is not 8, a different initial set is tried.

necessary to physically construct and count the two sets described in a problem.

Children often use two related Counting strategies to solve Join (Result Unknown) and Part-Part-Whole (Whole Unknown) problems. With *Counting On From First*, a child begins counting forward from the first addend in the problem. The sequence ends when the number of counting steps that represents the second addend has been completed. The following example illustrates this strategy:

> *Robin had 4 toy cars. Her friends gave her 7 more toy cars for her birthday. How many toy cars did she have then?*

> Jamie counts, "4 [pause], 5, 6, 7, 8, 9, 10, 11. She has 11 cars." As Jamie counts, he extends a finger with each count. When he has extended seven fingers, he stops counting and gives the answer. (See Figure 3.4.)

The *Counting On From Larger* strategy is identical to the Counting On From First strategy except that the child begins counting with the larger of the two addends. George uses this strategy in response to the problem posed above.

> George counts, "7 [pause], 8, 9, 10, 11—11 toy cars." George also moves his fingers as he counts, but the movement is very slight, and it is easy to miss his use of them to keep track.

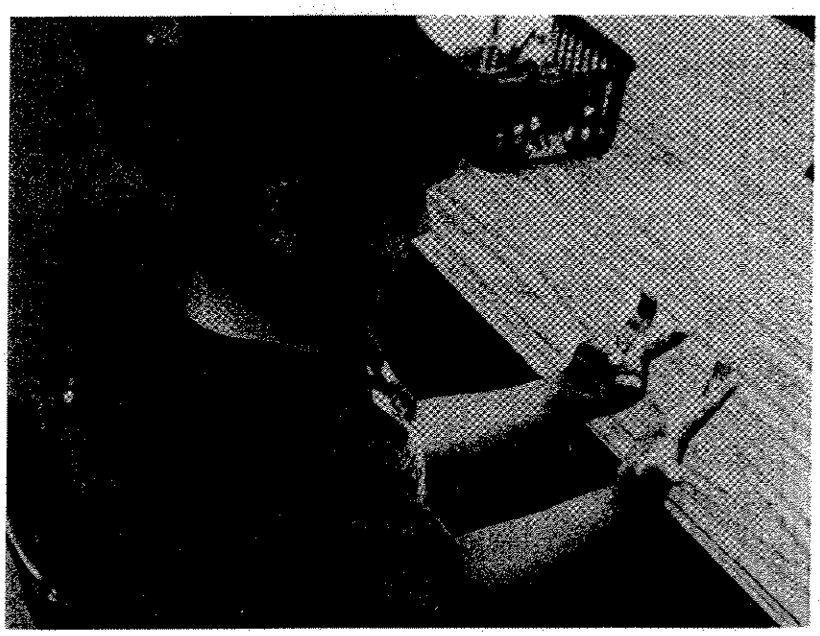

FIGURE 3.4
Using a Counting On From First Strategy to Solve a Join (Change Unknown) Problem

Note that in order to know when to stop counting, these two Counting strategies require some method of keeping track of the number of counting steps that represent the second addend. Most children use their fingers to keep track of the number of counts. A few may use counters or tallies, but a substantial number of children give no evidence of any physical action accompanying their counting. When counting is carried out mentally, it is difficult to determine how a child knows when to stop counting. Some children appear to use some sort of rhythmic or cadence counting such that counting words are clustered into groups of two or three. Others explicitly describe a double count (e.g., 6 is 1, 7 is 2, 8 is 3), but children generally have difficulty explaining this process.

When fingers or other objects are used in Counting strategies, they play a very different role than they do in Direct Modeling strategies. In this case, the fingers do not represent the second addend per se, but are used to keep track of the number of steps incremented in the counting sequence. When using fingers, children often do not appear to count their fingers; they recognize familiar finger patterns and can immediately tell when they have put up a given number of fingers.

A similar strategy is used to solve Join (Change Unknown) problems. Rather than the number reached being the answer, the answer is the number of steps in the counting sequence. The child initiates a forward counting strategy beginning with the smaller given number. The sequence ends with the larger given number. By keeping track of the number of counting words uttered in the sequence, the child determines the answer. This strategy, which is called *Counting On To*, is the counting analogue of the Direct Modeling strategy Joining To. The following example illustrates the Counting On To strategy:

> *Robin had 8 toy cars. Her parents gave her some more toy cars for her birthday. Then she had 13 toy cars. How many toy cars did her parents give her?*
>
> Ann counts, "8 [pause], 9, 10, 11, 12, 13." She extends a finger with each count as she says the sequence from 9 to 13. She looks at the extended fingers and responds, "They gave her 5."

Without counting, Ann could recognize that she had extended five fingers. Other children may have to actually count the extended fingers.

To reflect the action in the Separate (Result Unknown) problems, a backward counting sequence is employed. The child starts counting at the larger number given in the problem and counts backwards. This strategy, called *Counting Down*, is analogous to Separating From. Counting Down may take either of two forms:

> *Colleen had 11 guppies. She gave 3 guppies to Roger. How many guppies did she have left?*

> Ann counts, "11, 10, 9 [pause], 8. She had 8 left." Ann uses her fingers to keep track of the number of steps in the counting sequence.
>
> Bill counts, "11 [pause], 10, [raises one finger], 9 [raises a second finger], 8 [raises a third finger]. She had 8 left."

Ann says "11" as she mentally takes away the eleventh guppy, "10" as she takes away the tenth guppy, and "9" as she takes away the ninth guppy. The answer is the next (fourth) number in the backward sequence, 8. Bill's counting is different. As he takes one away, he says, "10," referring to the ten that remain and then "9" for the nine that remain. Finally, he says, "8" for the eight that remain as he removes the third guppy.

A backward counting sequence is also used to represent the action in a Separate (Change Unknown) problem. But the backward counting sequence in the *Counting Down To* strategy continues until the smaller number is reached; the number of words in the counting sequence is the solution to the problem. We illustrate this strategy, which is the counting counterpart of Separating To, below:

> *Colleen had 12 guppies. She gave some guppies to Roger. Then she had 8 guppies left. How many guppies did Colleen give to Roger?*
>
> Ann counts, "12 [extends one finger], 11 [extends a second finger], 10 [extends a third finger], 9 [extends a fourth finger and pauses] 8." She does not extend a finger for the 8. She looks at the 4 extended fingers and answers, "She gave 4 to Roger."
>
> Bill counts, "12 [pause], 11 [extends one finger], 10 [extends a second finger], 9 [extends a third finger], 8 [extends a fourth finger]." He looks at the 4 extended fingers and answers, "She gave 4 to Roger."

In Figure 3.5, we summarize the above Counting strategies.

DISTINCTION BETWEEN COUNTING AND MODELING STRATEGIES

It is important to note the distinction between Direct Modeling and Counting strategies. Direct Modeling is distinguished by the child's explicit physical representation of each quantity in a problem and the action or relationship involving those quantities before counting the resulting set. In using a Counting strategy, a child essentially recognizes that it is not necessary to actually construct and count sets. The answer can be figured out by focusing pg 20

FIGURE 3.5
Counting Strategies for Addition and Subtraction Problems

Problem	Strategy Description
Join (Result Unknown) Ellen had 3 tomatoes. She picked 5 more tomatoes. How many tomatoes does she have now?	*Counting On From First* The counting sequence begins with 3 and continues on 5 more counts. The answer is the last number in the counting sequence.
Join (Result Unknown) Ellen had 3 tomatoes. She picked 5 more tomatoes. How many tomatoes does she have now?	*Counting On From Larger* The counting sequence begins with 5 and continues on 3 more counts. The answer is the last number in the counting sequence.
Join (Change Unknown) Chuck had 3 peanuts. Clara gave him some more peanuts. Now Chuck has 8 peanuts. How many peanuts did Clara give to him?	*Counting On To* A forward counting sequence starts from 3 and continues until 8 is reached. The answer is the number of counting words in the sequence.
Separate (Result Unknown) There were 8 seals playing. 3 seals swam away. How many seals were still playing?	*Counting Down* A backward counting sequence is initiated from 8. The sequence continues for 3 more counts. The last number in the counting sequence is the answer.
Separate (Change Unknown) There were 8 people on the bus. Some people got off. Now there are 3 people on the bus. How many people got off the bus?	*Counting Down To* A backward counting sequence starts from 8 and continues until 3 is reached. The answer is the number of words in the counting sequence.

on the counting sequence itself. Counting strategies generally involve some sort of simultaneous double counting, and the physical objects a child may use (fingers, counters, tally marks) are used to keep track of counts rather than to represent objects in the problem.

Although children frequently use fingers with Counting strategies, the use of fingers does not distinguish Counting strategies from Direct Modeling strategies. As illustrated in the following examples (and shown in the photo at the beginning of Chapter 2), fingers may be used to directly model a problem or to keep track of the steps in a counting sequence:

> *Peter had 5 daisies. His sister gave him 3 more daisies. How many daisies did he have then?*
>
> Angela directly models: She puts up 3 fingers on one hand and 5 fingers on the other hand. She then counts her fingers, bending one slightly with each count, "1, 2, 3, 4, 5, 6, 7, 8. He has 8 daisies."
>
> Jerry uses his fingers in a counting strategy: He says, "5 [pause], 6, 7, 8," extending one finger for each count. "He has 8 daisies."

NUMBER FACTS

Children's solutions to word problems are not limited to Modeling and Counting strategies, as children do learn number facts both in and out of school and apply this knowledge to solve problems. Children learn certain number combinations before others, and they often use a small set of memorized facts to derive solutions for problems involving other number combinations. Children usually learn doubles (e.g., 4 + 4, 7 + 7) before other combinations, and they often learn sums of ten (e.g., 7 + 3, 4 + 6) relatively early. The following examples illustrate children's use of *Derived Facts:*

> *6 frogs were sitting on lily pads. 8 more frogs joined them. How many frogs were there then?*
>
> Rudy, Denise, Theo, and Sandra each answer, "14," almost immediately.
> *Teacher:* How do you know there were 14?
> *Rudy:* Because 6 and 6 is 12, and 2 more is 14.
> *Denise:* 8 and 8 is 16. But this is 8 and 6. That is 2 less, so it's 14.
> *Theo:* Well, I took one from the 8 and gave it to the 6. That made 7 and 7, and that's 14.
> *Sandra:* 8 and 2 more is 10, and 4 more is 14.

Derived Fact solutions are based on understanding relations between numbers, and it might be expected that they are used by only a handful of very bright students. This is not the case. Even without specific instruction, most children use Derived Facts before they have mastered all their number facts at a recall level. In a three-year longitudinal study of non-CGI classes, over 80 percent of the children used Derived Facts at some time in grades one through three, and Derived Facts represented the primary strategy of 40 percent of the children at some time during this period. In many CGI classes, most children use Derived Facts before they have learned all the number facts at a recall level. When children have the opportunity to discuss alternative strategies, the use of Derived Facts becomes even more prevalent.

Some children continue to use Counting strategies or Derived Facts for an extended period of time, and it should not be assumed that children recall facts simply because they appear to have recall of facts. Children can become very proficient in using Counting strategies and can apply them very quickly. Counting strategies and Derived Facts are relatively efficient strategies for solving problems. However, when given the opportunity to solve many problems with strategies they have invented, children eventually learn most number facts at a recall level.

RELATION OF STRATEGIES TO PROBLEM TYPES

We summarize the relation between strategies and problem types in Figure 3.6. Younger children generally select strategies that directly represent the action or relationships described in problems. For some problem types, the action dominates the problem more than others. Almost all children Join To

Problem Type	*Direct Modeling*	*Counting*
Join (Result Unknown) Part-Part-Whole (Whole Unknown)	Joining All	Counting On
Join (Change Unknown)	Joining To	Counting On To
Separate (Result Unknown)	Separating From	Counting Down
Separate (Change Unknown)	Separating To	Counting Down To
Compare (Difference Unknown)	Matching	**
Join (Start Unknown) Separate (Start Unknown)	Trial and Error	Trial and Error
Part-Part-Whole (Part Unknown)	**	**
Compare (Compared Quantity Unknown) Compare (Referent Unknown)	**	**

FIGURE 3.6
Relation of Strategy to Problem Types

*Note: ** indicates that there is not a commonly used strategy corresponding to the action or relationship described in the problem. For the Compare (Compared Quantity Unknown) problem, children usually use Joining All or Counting On strategies. For the other problems, children generally use Joining To, Separating From, Counting On To, or Counting Down.*

or Count On To to solve Join (Change Unknown) problems. Children consistently solve Separate (Result Unknown) problems with the Separating From strategy. Because it is relatively difficult to Count Down, some children use this strategy less frequently. As a consequence, some children who use Counting strategies that involve forward counting to solve other problems may continue to Separate From to solve Separate (Result Unknown) problems. Matching and Separating To are not used universally although most younger children use them. Children use Trial and Error even less frequently.

With experience in solving problems, children become more flexible in their selection of strategies so that they can select strategies that do not always correspond to the action in a given problem. However, even older children tend to select strategies that model the action or relationships for certain problems. Children generally model the very dominant action in a Separate (Change Unknown) or Join (Change Unknown) for an extended period of time. On the other hand, most older children replace the Matching strategy for Compare problems with one of the other common strategies that do not directly model the comparison relation described in Compare problems. As there is no counting analogue of Matching, children must select a strategy that does not model the Compare problem if they want to use a Counting strategy.

LEVELS OF DEVELOPMENT OF STRATEGIES

There is a great deal of variability in the ages at which children use different strategies. When they enter kindergarten, most children can solve some word problems using Direct Modeling strategies even when they have had little or no formal instruction in addition or subtraction. Some entering first-graders are able to use Counting strategies, and a few use Recall of Number Facts or Derived Facts consistently.

Most children pass through three levels in acquiring addition and subtraction problem-solving skills. Initially they solve problems exclusively by Direct Modeling. Over time, Direct Modeling strategies are replaced by the use of Counting strategies, and finally most children come to rely on number facts. The transition from Direct Modeling to using Counting strategies does not take place all at once, and for a time children may use both Direct Modeling and Counting strategies. Similarly, children learn a few number facts quite early, when they are still relying primarily on Direct Modeling or Counting strategies, and the use of recall and Derived Facts evolves over an extended period of time.

Direct Modeling Strategies

Initially, children are limited to Direct Modeling solutions. Direct modelers, however, are not uniformly successful in solving all problems that can be modeled because some problems are more difficult to model than others.

At first, young children are limited in their modeling capabilities. They do not plan ahead and can only think about one step at a time. This causes no problems with the Joining All and Separating From strategies, but it can cause difficulties in Joining To. Consider the following example:

Robin had 5 toy cars. How many more toy cars does she have to get for her birthday to make 9 toy cars?

Nick makes a set of 5 cubes and then adds 4 more cubes to the set counting, "6, 7, 8, 9," as he adds the cubes. He is not careful to keep the new cubes separate, so when he finishes adding them, he cannot distinguish them from the original set of 5 cubes. He looks confused for a moment and then counts the entire set of 9 cubes and responds, "9?"

Nick models the action in the problem, but he does not recognize that he needs to keep separate the four cubes that he adds on and the five cubes in the initial set. As a consequence, he has no way to figure out how many cubes he added. In other words, he simply modeled the action described in the problem without planning ahead how he was going to use his model to answer the question.

The only strategies available to children who only think about one step at a time are Joining All and Separating From. Therefore, they can only solve problems that can be modeled with these strategies: Join (Result Unknown), Part-Part-Whole (Whole Unknown), and Separate (Result Unknown) problems. With experience solving simple problems, children learn to reflect on their Modeling strategies. This gives them the ability to plan their solutions to avoid the errors illustrated in Nick's example above. Thus the ability to think about the entire problem—the question to be answered as well as the action in the problem—allows children to solve Join (Change Unknown) problems by Joining To.

Compare (Difference Unknown) problems may be slightly more difficult to model than Join (Change Unknown) problems. However, if the context or wording of the Compare problems provides cues for Matching, direct modelers can solve them. Graphing problems in which two quantities, like the number of boys and the number of girls in the class, are represented on a bar graph provide situations in which quantities can be compared. Children at this level quite readily solve such problems. The bar graph is a way of matching so that quantities can be compared in much the same way that the Matching strategy is applied to any Compare problem.

Most direct modelers have difficulty solving Start Unknown problems with understanding. Because the initial set is unknown, they cannot start out representing a given set. The only alternative for modeling the problem is Trial and Error, and Trial and Error is a difficult strategy for most direct modelers to apply. Direct modelers find the Part-Part-Whole (Part Unknown) problem difficult for a slightly different reason. Because there is no explicit action to represent, many of them have difficulty representing the problem with concrete objects. As a consequence, most direct modelers cannot solve this problem.

Counting Strategies

Gradually over a period of time, children replace concrete Direct Modeling strategies with more efficient Counting strategies, and the use of Counting strategies is an important marker in the development of number concepts. Counting strategies represent more than efficient procedures for calculating answers to addition and subtraction problems. They indicate a level of understanding of number concepts and an ability to reflect on numbers as abstract entities.

Initially, children may use both Direct Modeling and Counting strategies concurrently. At first, they use Counting strategies in situations in which they are particularly easy to apply, such as when the second addend is a small number or the first addend is relatively large:

> *Sam had 24 flowers. He picked 3 more. How many flowers did he have then?*

Even after children become quite comfortable with Counting strategies, they may occasionally fall back to Direct Modeling strategies with concrete objects. Most children come to rely on the Counting On strategies, but not all children use Counting Down consistently because of the difficulty in counting backwards.

Flexible Choice of Strategies

Initially, children use Counting strategies that are consistent with the action or relationships described in problems. In other words, the Counting strategies are abstractions of the corresponding Direct Modeling strategies they used previously. Over time, however, many children learn to represent problems with counting procedures that are not consistent with the structure of the problem. For example, they can solve Join (Start Unknown) problems by Counting On To, and they can solve Separate (Start Unknown) problems by Counting On. They also generally solve Compare (Difference Unknown) problems by either Counting Down or Counting On To, and some of them may even solve the Separate (Result Unknown) problems by Counting On To.

The development of understanding of part-whole relationships allows children to be more flexible in their choice of strategy. Children begin to learn that addition and subtraction problems can be thought of in terms of parts and wholes. In one large class of problems, the two parts are known, and the goal is to find the whole. In the other, one part and the whole are known, and the goal is to find the other part. All problems in which both parts are known can be solved by Joining All or Counting On. All problems in which one part and the whole are known can be solved using any one of several strategies, including Counting On To, Counting Down, Separating From, and so on. In this example, a child explains how she used part-whole relationships.

Some birds were sitting on a wire. 3 birds flew away. There were 8 birds still sitting on the wire. How many birds were sitting on the wire before the 3 birds flew away?

Kisha: [Counts] 8 [pause], 9, 10, 11. There were 11.
Teacher: I understand how you found the answer, but how did you know to count on like that?
Kisha: Well, there were the 3 birds that flew away, and the 8 birds that were still sitting there. And I want to know how many there were all together. The 3 and the 8 together were how many birds there were on the wire. So I put them together. I counted on from 8 to get 11.

For the Join (Result Unknown), Separate (Start Unknown), and Part-Part-Whole (Whole Unknown) problems, both parts are known. For the Separate (Result and Change Unknown), the Join (Change and Start Unknown), and the Part-Part-Whole (Part Unknown) problems, the whole and one of the parts are unknown. Because Compare problems involve two disjoint sets, the part-whole scheme does not apply well to these problems. But at about the same time that children develop an understanding of part-whole, they are able to be flexible in their choice of strategy for this type of problem also.

Another basic principle that allows children to be more flexible in dealing with Join and Separate problems is an understanding that actions can be reversed. In other words, the act of joining objects to a set can be undone by removing the objects from the resulting set. For example, the following Start-Unknown problem can be solved by reversing the action:

Colleen had some stickers. She gave 3 stickers to Roger. She had 5 stickers left. How many stickers did Colleen have to start with?

If the three stickers that were given away are put back with the five stickers that are left, the original set of stickers is restored. This analysis allows children to think of this problem in terms of a joining action so they can solve it without use of Trial and Error.

Number Facts

Even though Counting strategies can become very efficient, they are inefficient and distracting when dealing with very large numbers. Through experience solving problems, children begin to learn number facts so that they can recall them immediately. It is important to recognize, however, that number facts are learned at a recall level over a much longer period of time than previously has been assumed, and that many children may never learn all number facts at a recall level.

Children do not learn facts all at once, and they use selected number facts and Derived Facts at the direct modeling and counting levels. Although not all children use Derived Facts consistently, Derived Facts play an important role in solving problems and in learning number facts at a recall level. It is much easier for a child to learn to recall number facts if the child understands the relationships among number facts. Most children use Derived Facts for many combinations as they are learning number facts at a recall level, and it is important that all children at least understand relations among number facts (e.g., 6 + 7 is one more than 6 + 6), even if they do not use them consistently to derive other number facts.

INTEGRATION OF SOLUTION STRATEGIES AND PROBLEM TYPES

The relationships between strategies and problem types and the levels at which strategies may be used are represented in the Children's Solution Strategies chart (Figure 3.7). This figure presents a somewhat simplified version of what we have described above. Derived Facts and Recall of Number Facts are portrayed as cutting across all levels. Children use some number facts at all levels, and the use of number facts increases until number fact strategies become the dominant strategy.

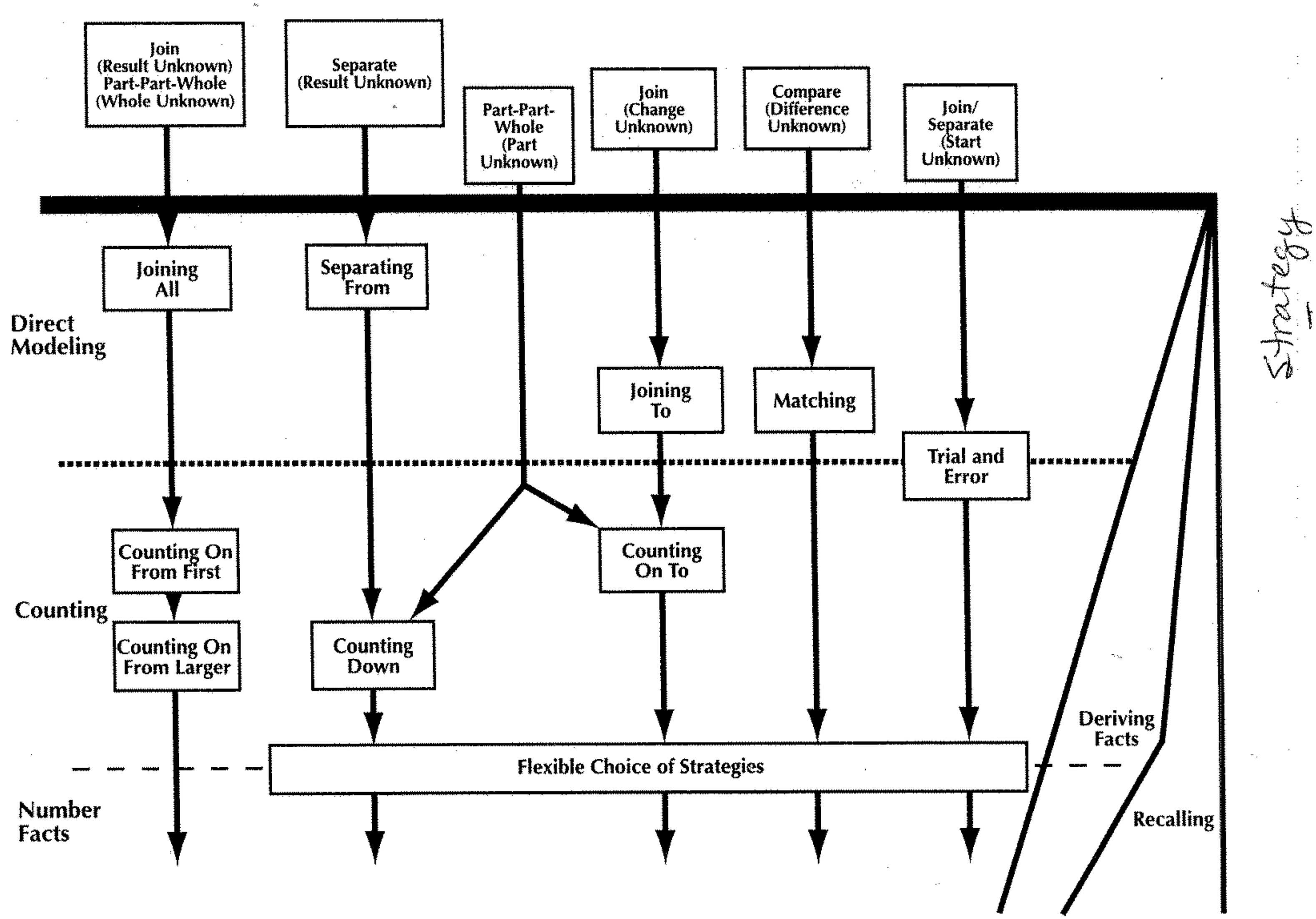

FIGURE 3.7 *Children's Solution Strategies*